Green's Functions and Numbering System for Transient Heat Conduction

James V. Beck*
Michigan State University, East Lansing, Michigan

A numbering system for transient heat conduction is proposed in connection with Green's functions. A tabulation of Green's functions for rectangular coordinates is given that can be used for a total of 220 different sets of boundary conditions; one-, two-, and three-dimensional cases are included. This total can be further extended for additional terms in the heat conduction equation. The proposed numbering system has a number of advantages, including the efficient tabulation and location of Green's functions and exact heat conduction solutions. Such a numbering system provides a framework for the computer cataloging and retrieval of a heat-transfer data base in diffusion and heat conduction.

Nomenclature

c = specific heat
f = nonhomogeneous boundary condition term
F = initial temperature distribution
g = volume energy source
G = Green's function
h = heat-transfer coefficient
k = thermal conductivity
m = coefficient in Eq. (7)
n = outward-pointing normal coordinate
r = radial coordinate
$\underline{r}$ = general coordinates
R = region
s = number of boundary conditions
s' = surface coordinate
S = surface
t = time
T = temperature
v' = volume coordinate
x,y,z = Cartesian coordinates
α = thermal diffusivity
δ = film thickness
θ,λ = angular spherical coordinates
ρ = density
ϕ = angular cylindrical coordinate

I. Introduction

A GREAT many exact solutions of linear transient heat conduction problems are in the literature. Notable compilation of solutions are contained in Carslaw and Jaeger,[1] Crank,[2] Schneider,[3] and Rohsenow and Hartnett.[4] There are many other solutions available in numerous books and journals. Indeed, one of the difficulties is that the location of a solution may be more time consuming than the rederivation.

Two objectives of this paper are 1) to provide a concise tabulation of Green's functions (GF) that will enable one to derive transient conduction solutions for rectangular coordinates efficiently and 2) to provide a numbering system that will enable efficient cataloging and locating of GF. A companion paper[5] provides a derivation of the Green's function equation, a discussion of methods of derivation of the GF themselves, and a very compact set of GF for the finite plate.

A number of GF for transient heat conduction for linear problems for homogeneous bodies of simple shapes are given in Ref. 1. Butkovskiy[6] gave a collection of GF for a variety of ordinary and partial differential equations, but the tables are not extensive for transient heat conduction. Reference 6 gives a numbering system for differential equations but does not provide one for boundary conditions. In transient heat conduction, the number of differential equations is small but the number of different cases generated by various boundary conditions is large. Hence, there is need for a numbering system that considers the boundary conditions. This paper provides such a numbering system, one that includes the rectangular, cylindrical, and spherical coordinate systems. This paper also includes some GF for the rectangular coordinate system, some early time forms of which are not included in Ref. 5 and others of which have not been previously given. Ozisik[7] gives an excellent discussion of the use of GF. The GF can be used with ideas[8] and programs[9] related to artificial intelligence.

One of the primary motivations of this work is the use of the GF in the unsteady surface element method.[10-18] It uses the GF as building blocks for much more complex geometries.

II. Equation and Boundary Conditions

The transient heat conduction equation can be written in the form

$$\nabla^2 T + \frac{1}{k} g(\underline{r},t) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \text{ in region } R \tag{1}$$

where T is temperature and $\underline{r}$ is the coordinate; k is thermal conductivity and α is thermal diffusivity, both of which are constants.

In this paper six boundary conditions (b.c.) are given. They are called the zeroth, first, ..., fifth kinds. The zeroth kind of boundary condition is for coordinate locations where there is no physical boundary; this condition occurs at $r_i = 0$ or $r_i \to \infty$ for radial coordinates and at $x \to \infty$ for rectangular coordinates.

The first kind of boundary condition is prescribed temperature (Dirichlet condition),

$$T(\underline{r}_i,t) = f_i(\underline{r},t) \text{ at surface } S_i \tag{2a}$$

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*Professor, Heat Transfer Group, Department of Mechanical Engineering. Member AIAA.

where the f_i function is arbitrary, including being equal to zero. The second kind of boundary condition involves the normal space derivative (also sometimes called the Neumann condition),

$$k\frac{\partial T}{\partial n_i} = f_i(\underline{r}_i,t) \text{ at surface } S_i \tag{2b}$$

where n_i is an outward-pointing normal and $f_i(\underline{r}_i,t)$ is the heat flux pointing *toward* the surface; that is, when $f_i(\cdot)>0$, the surface is heated. If $f_i(\cdot)=0$, the surface is said to have an insulation condition. The boundary condition of the third kind is the convective boundary condition and is given by

$$k\frac{\partial T}{\partial n_i} + h_i T = f_i(\underline{r}_i,t) \text{ at surface } S_i \tag{2c}$$

where h_i is the heat-transfer coefficient, n_i is the outward-pointing normal, and $f_i(\underline{r}_i,t)$ could be the product of h_i and an ambient temperature T_∞. The heat-transfer coefficient is a constant for a given surface, but $f_i(\underline{r}_i,t)$ can be a function of surface location and time. The above three boundary conditions have long been referred to as the first, second, and third kinds, but the fourth and fifth conditions have not been so agreed upon. The fifth condition is taken to be

$$k\frac{\partial T}{\partial n_i} + h_i T = f_i(\underline{r}_i,t) + (\rho c\delta)_i\frac{\partial T}{\partial t} \text{ at surface } S_i \tag{2d}$$

where $(\rho c\delta)$ is the heat capacity of a surface film or well-stirred fluid. When $h_i=0$ in Eq. (2d), the fourth boundary condition is obtained. The initial temperature distribution is given by

$$T(\underline{r},0) = F(\underline{r}) \tag{2e}$$

The Green's function $G(\underline{r},t|\underline{r}',\tau)$ is a solution to Eq. (1), with g being the impulse function and with homogeneous boundary conditions, i.e., the f_i functions in Eq. (2) are set equal to zero. The GF solution for the temperature for nonzero f_i values (nonhomogeneous boundary conditions) and a nonzero g is given by[5]

$$T(\underline{r},t) = \int_R G(\underline{r},t|\underline{r}',0)F(\underline{r}')\mathrm{d}v$$

$$+\frac{\alpha}{k}\int_{\tau=0}^{t}\mathrm{d}\tau\int_R g(\underline{r}',\tau)G(\underline{r},t|\underline{r}',\tau)\mathrm{d}v'$$

$$+\alpha\int_{\tau=0}^{t}\mathrm{d}\tau\sum_{i=1}^{s}\int_{S_i}\frac{f_i(\underline{r}_i',\tau)}{k}G(\underline{r},t|\underline{r}_i',\tau)\mathrm{d}s_i'$$

(b.c. of the 2nd, 3rd, 4th, and 5th kinds)

$$+\alpha\int_{\tau=0}^{t}\mathrm{d}\tau\sum_{i=1}^{s}\int_{S_i}f_i(\underline{r}_i',\tau)\frac{\partial G}{\partial n_i'}\bigg|_{\underline{r}'=\underline{r}_i'}\mathrm{d}s_i'$$

(b.c. of the 1st kind only)

$$+\alpha\sum_{i=1}^{s}\int_{S_i}\frac{(\rho c\delta)_i}{k}G(r,t|\underline{r}_i',0)F(\underline{r}_i')\mathrm{d}s_i' \tag{3}$$

(b.c. of the 4th and 5th kinds)

Boundary conditions of the zeroth kind do not need to be explicitly included in Eq. (3).

III. Numbering System

Rectangular Coordinates, One-Dimensional Case

For the case of rectangular coordinates, the one-dimensional (1-D) transient heat conduction equation can be written as

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{k}g(x,t) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4}$$

where x could also be y or z. In the boundary conditions given by Eqs. (2b-d), the outward-pointing coordinate n_i at $x=0$ is replaced by $-x$ and at $x=L_x$ by x.

For any 1-D Green's function case in rectangular coordinates, the numbering system is to start with a capital X

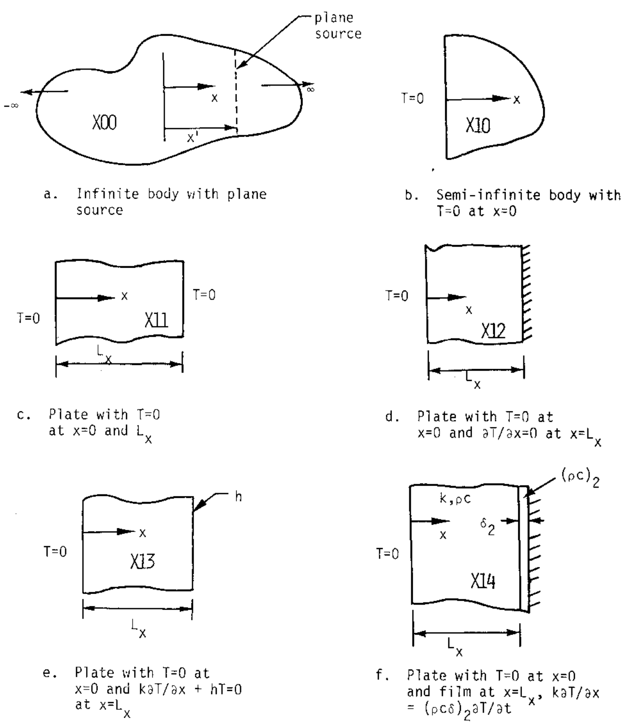

Fig. 1 Some geometries and boundary conditions for rectangular coordinates.

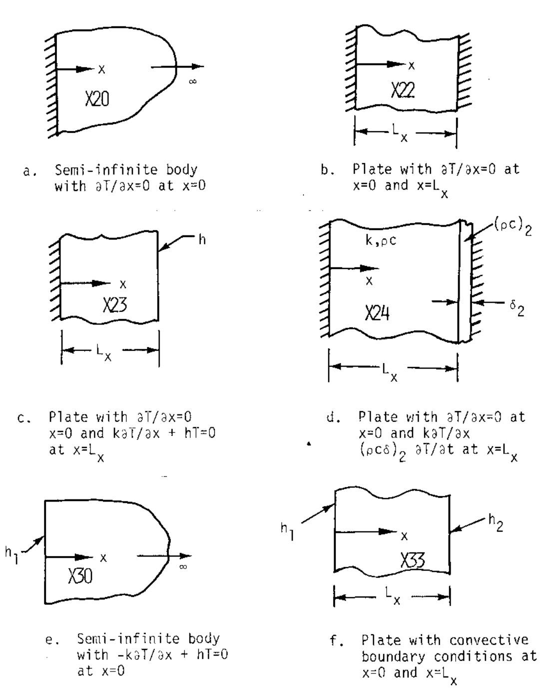

Fig. 2 More boundary conditions for rectangular coordinates.

followed by two digits; the first digit is for the first boundary condition (located at the smallest x coordinate in the problem), and the second for the other boundary condition. These digits vary from 0 to 5, corresponding to the boundary condition kind.

Some possible geometries and boundary conditions are shown in Figs. 1 and 2. Figure 1a is for an infinite body, so its number is X00; there is no physical boundary at $x = \pm\infty$. An instantaneous *plane* source is implied by the one-dimensional conditions as shown in Fig. 1a. Figure 1b is for a semi-infinite body with $T=0$ at $x=0$. The $T=0$ condition is a boundary condition of the first kind so that the digit 1 is associated with the first boundary condition. Since there is no physical boundary as $x \to \infty$, the second boundary condition is of the zeroth kind. Then Fig. 1b has the number X10. In a similar fashion, the plate cases in Figs. 1c-f are numbered X11, X12, X13, and X14. Additional boundary conditions are shown in Fig. 2. The GF associated with these cases are given in the Appendix.

A complete set of numbers for the 1-D rectangular coordinates is the following set of 31 numbers:

X00

X10 X11 X12 X13 X14 X15

X20 X21 X22 X23 X24 X25

X30 X31 X32 X33 X34 X35

X40 X41 X42 X43 X44 X45

X50 X51 X52 X53 X54 X55 (5)

In a table of Green's functions it is not necessary to tabulate all of these because the XJI function can be obtained from XIJ, provided both I and J are 1, 2, 3, 4, or 5; XJI is obtained from XIJ by replacing x with $L_x - x$. If these redundant numbers are eliminated, 21 cases remain. If only boundary conditions of the zeroth, first, second, and third kinds are considered, there are 10 basic cases.

Rectangular Coordinates, Two- and Three-Dimensional Cases

For all the boundary conditions a numbering system for two- and three-dimensional GF can be generated. For the two-dimensional (2-D) case, one can write XIJYKL, where IJ (associated with X) and KL (associated with Y) are any of the combinations of two numbers given in Eq. (5). For example, for the 2-D infinite regions, one would have X00Y00, which would correspond to an instantaneous line source in an infinite medium. Some other two-dimensional cases are shown in Figs. 3a-e. The geometries vary from a semi-infinite region to a two-dimensional plate. Even though a geometry does not impose two-dimensional conditions as depicted in Figs. 3b and 3c, the initial temperature distribution or the nonhomogeneous boundary conditions ($f_i \neq 0$) can cause the solution for T to be 2-D and thus require 2-D Green's functions.

The number of distinct cases can be calculated from

$$\binom{m+n-1}{n} = \frac{(m+n-1)!}{n!\,(m-1)!} \tag{6}$$

where m is the number of distinct 1-D cases and n is the number of dimensions. For $m=21$ and $n=2$, the number of distinct 2-D cases is 231. If the boundary conditions of the fourth and fifth kinds are eliminated, there are 10 distinct 1-D cases that can be used to obtain 55 distinct 2-D cases. Green's functions for the boundary conditions of the zeroth through the third kinds for 1-D cases can be utilized by using simple multiplication to obtain the 55 2-D GF.[5]

One example of a three-dimensional (3-D) problem is illustrated by Fig. 3f. A point source in a 3-D infinite body has the number X00Y00Z00. If the boundary conditions of the fourth and fifth kinds are excluded, one uses $m=10$ and $n=3$ to obtain from Eq. (6) the value of 220 cases. The Green's functions for these 220 cases can be simply found by multiplication of the 1-D GF.[5] Consequently, the 10 distinct 1-D cases can be readily used to obtain a total of $10+55+220=285$ Green's functions. Actually, one can obtain considerably more cases, as discussed next.

Another related differential equation is

$$\frac{\partial^2 T}{\partial x^2} - U\frac{\partial T}{\partial x} - m^2 T + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{7}$$

where $U(\partial T/\partial x)$ can model a uniform flow and $-m^2 T$ can simulate side heat losses. By using the transformation[7]

$$T(x,t) = W(x,t)\exp\left[\frac{Ux}{2} - \left(\frac{U^2}{4} + m^2\right)\alpha t\right] \tag{8}$$

Eq. (7) is replaced by

$$\frac{\partial^2 W}{\partial x^2} + \frac{g}{k}\exp\left[-\frac{Ux}{2} + \left(\frac{U^2}{4} + m^2\right)\alpha t\right] = \frac{1}{\alpha}\frac{\partial W}{\partial t} \tag{9}$$

The Green's function solution for $T(x,t)$ for boundary conditions of the zeroth to third kinds is obtained using Eq. (3),

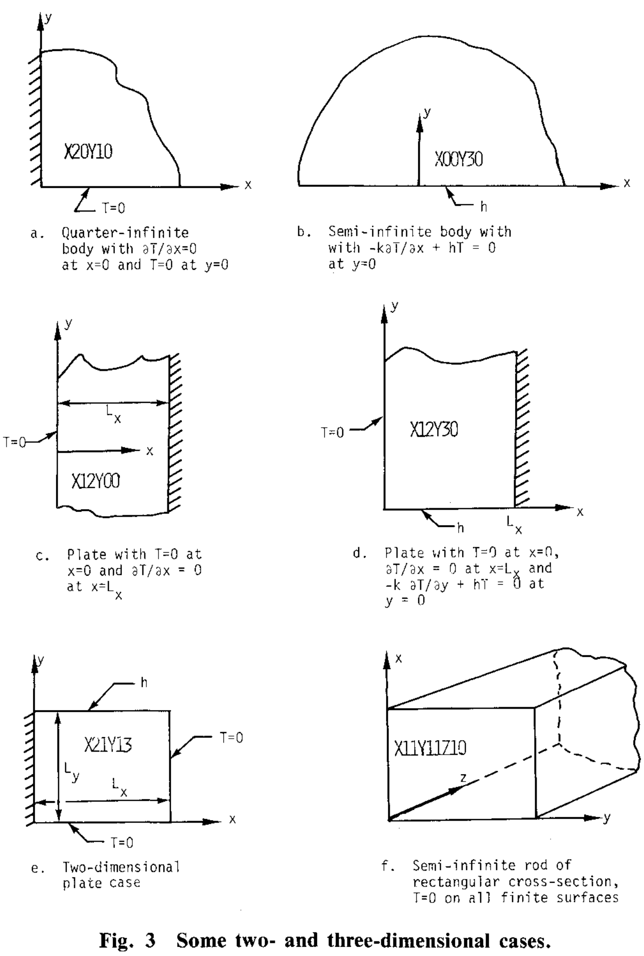

Fig. 3 Some two- and three-dimensional cases.

$$T(x,t)=\exp\left[\frac{Ux}{2}-\left(\frac{U^2}{4}+m^2\right)\alpha t\right]$$

$$\times\left\{\int_R G(x,t|x',0)F(x')e^{-Ux'/2}\mathrm{d}x'\right.$$

$$+\frac{\alpha}{k}\int_{\tau=0}^{t}\int_{x_1'}^{x_2'} g(x',\tau)G(x,t|x',\tau)$$

$$\times\exp\left[-\frac{Ux'}{2}+\left(\frac{U^2}{4}+m^2\right)\alpha\tau\right]\mathrm{d}x'\mathrm{d}\tau$$

$$+\alpha\int_{\tau=0}^{t}\left[\frac{f_1(x_1',\tau)}{k}G(x,t|x_1',\tau)e^{-(Ux_1'/2)}\right.$$

$$\left.+\frac{f_2(x_2',\tau)}{k}G(x,t|x_2',\tau)e^{(-Ux_2')/2}\right]\exp\left[\left(\frac{U^2}{4}+m^2\right)\alpha\tau\right]\mathrm{d}\tau$$

(b.c. of the 2nd and 3rd kinds only)

$$+\alpha\int_{\tau=0}^{t}\left[f_1(x_1',\tau)\frac{\partial G}{\partial x'}\bigg|_{x'=x_1'}e^{-Ux_1'/2}\right.$$

(b.c. of the 1st kind only)

$$\left.\left.-f_2(x_2',\tau)\frac{\partial G}{\partial x'}\bigg|_{x'=x_2'}e^{-Ux_2'/2}\right]\exp\left[\left(\frac{U^2}{4}+m^2\right)\alpha\tau\right]\mathrm{d}\tau\right\} \quad (10)$$

where x_1' and x_2' are the boundaries and $x_1'<x_2'$.

The Green's functions are found from the Appendix with the following changes:

1) Zeroth and first boundary conditions: none.

2) Second and third boundary conditions: use the GF for the *third* kind with h replaced by $h-0.5Uk$ at $x=x_1'$ and $h+0.5Uk$ at $x=x_2'$ ($h=0$ for second kind of boundary condition).

Hence, by utilizing these relations, the GF in the Appendix can be applied to a more general equation than originally intended. Furthermore, it is not necessary to compile GF separately for all these cases as is attempted in Butkovskiy.[6] In addition, these 1-D GF can be used for 2-D and 3-D forms of the rectangular coordinate case given by Eq. (7); this is done by using simple multiplicaton of 1-D Green's functions (excluding the boundary conditions of the fourth and fifth kinds[5]).

Cylindrical Coordinates: Radial Cases

The cylindrical, radial, transient heat conduction equation can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (11)$$

and GF can be given for a number of cases. Because of space limitations, none are listed herein but some are given in Carslaw and Jaeger[1] and Butkovskiy.[6] Nevertheless, a numbering system is suggested.

The numbers start with R and are followed by two digits as for the rectangular system. A complete list is

R00	R01	R02	R03	R04	R05
R10	R11	R12	R13	R14	R15
R20	R21	R22	R23	R24	R25
R30	R31	R32	R33	R34	R35
R40	R41	R42	R43	R44	R45
R50	R51	R52	R53	R54	R55

(12)

which includes 36 cases. R00 is for an annular source in an infinite medium. R10, R20, R30, R40, and R50 are for the infinite region *outside* a circular hole, whereas R01,...,R05 are for infinite solid cylinders. These 10 cases are quite different and there is no analogous situation in the rectangular coordinate system. The remaining cases are for infinite hollow cylinders. There is limited repetition since R21 is similar to R12 and so on; this means that there are 26 distinct cases. If the fourth and fifth kinds of boundary condition cases are dropped, there are 13 distinct cases left.

The one-dimensional axial coordinate in radial coordinates, sometimes called x, is exactly the same as the 1-D rectangular coordinate. As discussed in Ref. 5, the Green's functions for the (r,x) coordinates can be simply obtained by multiplying the 1-D GF for r and x together (for all the boundary conditions except the fourth and fifth). The geometries covered include finite, semi-infinite, and infinite cylinders; annuli; and solids. Since there are 10 possible distinct x cases (excluding the fourth and fifth boundary conditions) and 13 for the r coordinates, there are 130 cases that can be readily found by multiplication and, hence, do not need to be separately tabulated. A number for a sample case is R12X30 for which the Green's function can be found by multiplying the GF for R12 and X30 together, for example.

Cylindrical Coordinates—Angular Cases

The one-dimensional angular equation for cylindrical coordinates is

$$\frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (13)$$

For a problem to be only a function of ϕ, it is necessary that r be a constant, i.e., a radially thin section is being considered. The proposed numbering system for this coordinate system starts with Φ and a complete set of cases is

Φ00				
Φ11	Φ12	Φ13	Φ14	Φ15
Φ21	Φ22	Φ23	Φ24	Φ25
Φ31	Φ32	Φ33	Φ34	Φ35
Φ41	Φ42	Φ43	Φ44	Φ45
Φ51	Φ52	Φ53	Φ54	Φ55

(14)

The Φ00 case is for a complete annulus as shown by Fig. 4a, and all the other cases are open over some angle as shown by Fig. 4b. The only special case is Φ00, since the remaining cases are the same as for XIJ system (I,J=1,...,5) with $r\phi$ replacing x in Eq. (13).

Two-dimensional cases with x and ϕ can also be formed with the product of the 1-D Green's functions. One could have, for example, Φ00X11, which can be found using the product of the 1-D GF for Φ00 and X11 (see Fig. 4c).

The 2-D cases for r and ϕ can be typically written as R01Φ23 which is shown in Fig. 4d. The GF for the r,ϕ coordinates must be obtained from 2-D solutions. If the fourth and fifth boundary conditions are not included, there are a total of 91 distinct cases (13 times 7).

Three-dimensional GF cases formed for r,ϕ,x can be written as R01Φ23X11, for example; the Green's function can be obtained using the product of the R01Φ23 and X11 GF.

Spherical Coordinates Cases

The transient radial, spherical heat conduction equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (15)$$

The geometries for the Green's function cases for this equation are similar to those for cylindrical coordinates. The first letter is now replaced by RS so that Eq. (14) starts with RS00, RS10, etc. The RS00 case is for an instantaneous spherical shell source which is different from the Green's function for a point source, X00Y00Z00.

The coordinates for three-dimensional spherical cases can be taken to be (r,θ,λ), where $0<\theta<\pi$ and $0\leq\lambda\leq 2\pi$. One case for a solid sphere would be RS01Θ00Λ00. No multiplication of the 1-D GF for spherical coordinates is possible.

IV. Discussion of Green's Function Values

Green's functions for various boundary conditions and coordinate systems can be quite different in numerical values and shape. Even so, a number of general statements can be made.

First, the 1-D GF are functions of four quantities, such as x, x', t, and τ. Fortunately, t and τ can be replaced by $t-\tau$ so that there is actually only a function of three variables. This is one more than for temperature itself.

Second, the GF are always equal to or greater than zero because they are the results of heat *sources* (rather than sinks).

Third, sometimes two different expressions, such as given for case X11, are available. The ones involving the trigonometric relations sometimes yield T results that require fewer terms for evaluation for "large times" than the GF involving only exponentials, which are better for small times. For the X11, X12, and X22 cases, only a few terms are needed if the proper series is chosen. It is usually computationally advantageous to use both series expressions for a given solution.

Table 1 Number of distinct cases for rectangular coordinates for boundary conditions kinds 0, 1, 2, and 3

Row	Coordinate(s)	No. of distinct cases	No. of special cases
1	x,y or z	10	10
2	(x,y) or (x,z) or (y,z)	55	10[a]
3	(x,y,z)	220	10[a]

[a]Same cases as included in row 1.

Table 2 Number of distinct cases for cylindrical coordinates for boundary conditions kinds 0, 1, 2, and 3

Row	Coordinate(s)	No. of distinct cases	No. of special cases
1	r	13	13
2	ϕ	7	7[a]
3	x	10	10[b]
4	(r,x)	130	23[c]
5	(r,ϕ)	91	91
6	(ϕ,x)	70	17[d]
7	(r,ϕ,x)	910	101[e]

[a]Only one new case. Others included in Table 1, row 1. [b]Same cases as in Table 1, row 1. [c]Same cases included in Table 2, rows 1 and 3. [d]Same cases included in Table 2, rows 2 and 3. [e]Same cases included in Table 2, rows 3 and 5.

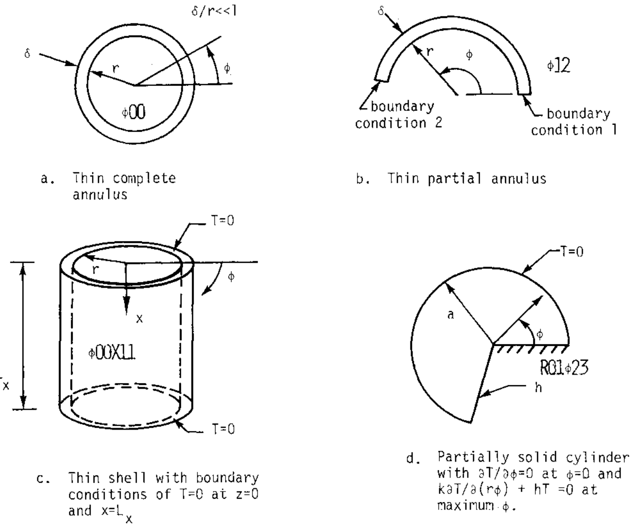

Fig. 4 Some angular geometry cases.

V. Summary and Conclusions

A numbering system is given for Green's functions for the linear, transient heat conduction equation. The *same* numbering system is proposed for transient heat conduction solutions, whether or not GF are available. Rectangular, cylindrical, and spherical coordinates are covered. The total numbers of possible cases for rectangular and cylindrical coordinates are summarized in Tables 1 and 2 for boundary condition of the zeroth to the fourth kind. Only 10 1-D Green's functions need be derived for rectangular coordinates; by using the multiplication principle, these ten distinct cases can be directly employed to obtain 55 2-D cases and 220 3-D cases or a total of 285 cases. For cylindrical coordinates the number of distinct cases is given in Table 2. For some of these cases the GF are the same as those found in Table 1. The greatest number of special cases comes from the 2-D problems involving r and ϕ coordinates. If the GF found for Table 1 are excluded, there are 13 new cases for the r coordinate, 1 for ϕ, and 91 for (r,ϕ), or a total of 105 special GF. Utilizing these GF and those in Table 1, a grand total of 1516 cases is included in Tables 1 and 2, but only 115 special GF need be derived and most, 91, are for (r,ϕ).

The boundary conditions of the fourth and fifth kinds are also discussed. They involve a film of finite heat capacity. One-dimensional GF for this case are given in the Appendix, but unfortunately multiplication of GF with this boundary condition is not possible. Green's functions for cases X40 and X50 have not been given previously.

The Green's functions considered are for the usual transient heat conduction equation,

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

and it is also shown how the same GF can be used for additional terms in the heat conduction equation. One of the additional terms could simulate side heat losses, and another term could flow through the body.

The numbering system provided herein can be incorporated into a computerized information bank on heat transfer.[8] This system can aid in organizing and retrieving the needed GF or heat-transfer solutions. In the future it may be possible to use a symbolic algebraic computer program[9] to utilize the Green's functions to generate mathematical expressions, to evaluate these expressions, and to plot the results. Such a possiblity is very important and can be considered to be computer-aided engineering in heat transfer.

The proposed numbering system for transient heat conduction is also applicable to mass transfer and flow in porous media. Moreover, it can be used for steady-state convection correlations.

Appendix

X00 Infinite body with an instantaneous plane source (see Fig. 1a)

$$G(x,t|x',\tau)=[4\pi\alpha(t-\tau)]^{-1/2}\exp\left[-\frac{(x-x')^2}{4\alpha(t-\tau)}\right]$$

X10 Semi-infinite region with $T=0$ at $x=0$ (see Fig. 1b)

$$G(x,t|x',\tau)=\frac{1}{(4\pi\alpha(t-\tau))^{1/2}}\left[\exp\left(-\frac{(x-x')^2}{4\alpha(t-\tau)}\right)-\exp\left(-\frac{(x-x')^2}{4\alpha(t-\tau)}\right)\right]$$

X11 Plate with $T=0$ at $x=0$ and L_x (see Fig. 1c)

For $\alpha(t-\tau)/L^2>0.2$ and for $m\leq 2$, the error is less than $(4E-8)/L_x$ using

$$G(x,t|x',\tau)=\frac{2}{L_x}\sum_{m=1}^{\infty}\exp\left[-m^2\pi^2\frac{\alpha(t-\tau)}{L_x^2}\right]\times\sin\left(m\pi\frac{x}{L_x}\right)\sin\left(m\pi\frac{x'}{L_x}\right)$$

For $\alpha(t-\tau)/L_x^2<0.2$ and for $|n|\leq 2$, the error is less than $(2E-14)/L_x$ using

$$G(x,t|x',\tau)=[4\pi\alpha(t-\tau)]^{-1/2}\times\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(2nL_x+x-x')^2}{4\alpha(t-\tau)}\right]-\exp\left[-\frac{(2nL_x+x+x')^2}{4\alpha(t-\tau)}\right]\right\}$$

X12 Plate with $T=0$ at $x=0$ and $\partial T/\partial x=0$ at $x=L_x$ (see Fig. 1d)

$$G(x,t|x',\tau)=\frac{2}{L_x}\sum_{m=1}^{\infty}\exp\left[-\beta_m^2\frac{\alpha(t-\tau)}{L_x^2}\right]\times\sin\left(\beta_m\frac{x}{L_x}\right)\sin\left(\beta_m\frac{x'}{L_x}\right)$$

$$\times\beta_m=(2m-1)\pi/2,\qquad m=1,2,3,\ldots$$

An alternative form is

$$G(x,t|x',\tau)=[4\pi\alpha(t-\tau)]^{1/2}\sum_{n=-\infty}^{\infty}(-1)^n\times\left\{\exp\left[-\frac{(2nL_x+x-x')^2}{4\alpha(t-\tau)}\right]-\exp\left[-\frac{(2nL_x+x+x')^2}{4\alpha(t-\tau)}\right]\right\}$$

(see Ref. 1, p. 275).

X13 Plate with $T=0$ at $x=0$ and convective boundary conditions at $x=L_x$ (see Fig. 1e)

$$G(x,t|x',\tau)=\frac{2}{L_x}\sum_{m=1}^{\infty}\exp\left[-\beta_m^2\frac{\alpha(t-\tau)}{L_x^2}\right]\times\frac{(\beta_m^2+B^2)\sin[\beta_m(x/L_x)]\sin[\beta_m(x'/L_x)]}{\beta_m^2+B^2+B}$$

Eigencondition: $\beta_m\cot(\beta_m)=-B,\ B\equiv hL_x/k$

X20 Semi-infinite body with $\partial T/\partial x=0$ at $x=0$ (see Fig. 2a)

$$G(x,t|x',\tau)=\frac{1}{[4\pi\alpha(t-\tau)]^{1/2}}\times\left[\exp\left(-\frac{(x-x')^2}{4\alpha(t-\tau)}\right)+\exp\left(-\frac{(x+x')^2}{4\alpha(t-\tau)}\right)\right]$$

X22 Plate with $\partial T/\partial x=0$ at $x=0$ and L_x (see Fig. 2b)

$$G(x,t|x',\tau)=\frac{1}{L_x}\left[1+2\sum_{m=1}^{\infty}\exp\left[-m^2\pi^2\frac{\alpha(t-\tau)}{L_x^2}\right]\times\cos\left(\frac{m\pi x}{L_x}\right)\cos\left(\frac{m\pi x'}{L_x}\right)\right]$$

An alternative form is

$$G(x,t|x',\tau)=[4\pi\alpha(t-\tau)]^{-1/2}\times\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(2nL_x+x-x')^2}{4\alpha(t-\tau)}\right]+\exp\left[-\frac{(2nL_x+x+x')^2}{4\alpha(t-\tau)}\right]\right\}$$

(see Ref. 1, p. 275).

X23 Plate with $\partial T/\partial x=0$ at $x=0$ and convective boundary condition $x=L_x$ (see Fig. 2c)

$$G(x,t|x',\tau)=\frac{2}{L_x}\sum_{m=1}^{\infty}\frac{\beta_m^2+B^2}{\beta_m^2+B^2+B}\times\exp\left[-\frac{\alpha(t-\tau)\beta_m^2}{L_x^2}\right]\cos\left(\beta_m\frac{x}{L_x}\right)\cos\left(\beta_m\frac{x'}{L_x}\right)$$

Eigencondition: $\beta_m\tan(\beta_m)=B,\ B\equiv hL_x/k$

X30 Semi-infinite body with convective boundary condition (Fig. 2e)

$$G(x,t|x',\tau)=[4\pi\alpha(t-\tau)]^{-1/2}\left\{\exp\left[-\frac{(x-x')^2}{4\alpha(t-\tau)}\right]+\exp\left[-\frac{(x+x')^2}{4\alpha(t-\tau)}\right]\right\}$$

$$-\frac{h}{k}\exp[\alpha(t-\tau)h^2k^{-2}+h(x+x')k^{-1}]\times\operatorname{erfc}\left[\frac{x+x'}{(4\alpha(t-\tau))^{1/2}}+\frac{h}{k}(\alpha(t-\tau))^{1/2}\right]$$

(Ref. 1, p. 358).

X33 Plate with convective boundary conditions at both surfaces (Fig. 2f)

$$G(x,t|x',\tau)=\frac{2}{L_x}\sum_{m=1}^{\infty}\exp\left[-\beta_m^2\frac{\alpha(t-\tau)}{L_x^2}\right]\times\left\{\left[\beta_m\cos\left(\beta_m\frac{x}{L_x}\right)+B_1\sin\left(\beta_m\frac{x}{L_x}\right)\right]\times\left[\beta_m\cos\left(\beta_m\frac{x'}{L_x}\right)+B_1\sin\left(\beta_m\frac{x'}{L_x}\right)\right]\right\}$$

$$\div\left\{(\beta_m^2+B_1^2)\left(1+\frac{B_2}{\beta_m^2+B_2^2}\right)+B_1\right\}$$

where β_m are the positive eigenvalues (arranged in increasing order) of

$$\tan\beta_m = \frac{\beta_m(B_1+B_2)}{\beta_m^2 - B_1B_2}, \qquad B_1 \equiv \frac{h_1L_x}{k}, \qquad B_2 \equiv \frac{h_2L_x}{k}$$

X40 Semi-infinite body with $-k\partial T/\partial x + (\rho c\delta)_1 \partial T/\partial t = 0$ at $x=0$

$$G(x,t|x',\tau) = G_{X10} + \frac{1}{\delta_1 P}\exp\left[-\frac{(x+x')^2}{4\alpha(t-\tau)}\right]$$

$$\times \operatorname{rerf}\left[\frac{x+x'}{2(\alpha(t-\tau))^{1/2}} + \frac{1}{P}\frac{(\alpha(t-\tau))^{1/2}}{\delta_1}\right]$$

$$P \equiv \frac{(\rho c)_1}{\rho c}, \qquad \operatorname{rerf}(z) \equiv \exp(z^2)\operatorname{erfc}(z)$$

X50 Semi-infinite body with $-k\partial T/\partial x + hT + (\rho c\delta)_1 \partial T/\partial t = 0$ at $x=0$

$$G(x,t|x',\tau) = G_{X10} + \frac{1}{2\delta_1 AP}\exp\left[-\frac{(x+x')^2}{4\alpha(t-\tau)}\right]$$

$$\times \left\{(1+A)\operatorname{rerf}\left[\frac{x+x'}{2(\alpha(t-\tau))^{1/2}} + (1+A)\frac{(\alpha(t-\tau))^{1/2}}{2\delta_1 P}\right]\right.$$

$$\left. -(1-A)\operatorname{rerf}\left[\frac{x+x'}{2(\alpha(t-\tau))^{1/2}} + (1-A)\frac{(\alpha(t-\tau))^{1/2}}{2\delta_1 P}\right]\right\}$$

$$A \equiv (1-4BP)^{1/2} \text{ for } 4BP<1$$

$$B \equiv h\delta_1/k$$

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